VI Characteristic Foliations

by Th^{m} II.5 we know the characteristic foliation on a surface determines the "germ" of a contact structure

here we study characteristic foliations more carefully

(why are surfaces so important?
can cut a contact manifold into simple pieces using surfaces. If you understand the contact structure on the pieces and near the surfaces you might understand contact str on original manifold)

let Σ be a surface
ω an area form on Σ
v a vector field on Σ

the divergence of v is the function $\mathrm{div}_\omega v$ satisfying

$$\mathcal{L}_v \omega = (\mathrm{div}_\omega v)\,\omega$$

or equivalently

$$d\iota_v\omega = (\mathrm{div}_\omega v)\,\omega$$

note:

1) if x is a singular point of v (i.e. $v(x)=0$), then the divergence is independent of ω (def^{n} same orientation)

indeed let $\omega' = e^f\omega$ ← any positive function can be written this way

$$d\iota_v\omega' = d(e^f\iota_v\omega) = e^f(df\wedge\iota_v\omega + d\iota_v\omega)$$
$$\overset{*}{=} e^f(df(v)\,\omega + d\iota_v\omega)$$
$$= (df(v) + \mathrm{div}_\omega v)\,\underbrace{e^f\omega}_{\omega'}$$

3 form on Σ so 0

$$*\ \iota_v(\overbrace{df\wedge\omega}) = 0$$
$$\| $$
$$df(v)\,\omega - df\wedge\iota_v\omega$$

so $div_{\omega'} v = df(v) + div_\omega v$

if $v(x) = 0$ then $df(v(x)) = 0$ so at x

$$div_{\omega'} v = div_\omega v$$

2) let's compute $div_\omega fv$

$$d\iota_{fv}\omega = d(f\iota_v\omega) = df \wedge \iota_v\omega + f\, d\iota_v\omega$$
$$= (df(v) + f\, div_\omega v)\,\omega$$

so if x a singular point of v and $div_\omega v = 0$ at x

then $div_\omega v' = 0$ at x for any rescalling v' of v

$$\boxed{\text{so it makes sense to talk about whether or not a singular point in a } \underline{\text{singular foliation}} \text{ has zero divergence!}}$$

Prop 1:

let $\mathcal{F}$ be a singular foliation on an oriented surface Σ

$\mathcal{F}$ is the characteristic foliation induced on Σ by some contact structure

$\iff$

all the singular points of $\mathcal{F}$ have non-zero divergence

Proof: $(\Rightarrow)$

on $\Sigma \times [-1,1]$ with t variable on $[-1,1]$

consider a contact 1-form

$$\alpha = \beta_t + u_t\, dt$$

for β_t a 1-form on Σ
u_t a function on Σ

here $\Sigma = \Sigma \times \{0\}$

$\Sigma_\xi = \ker \beta_0$

$$d\alpha = \underset{\substack{\uparrow \\ d \text{ on } \Sigma}}{d_2}\beta_t + dt \wedge \frac{\partial \beta_t}{\partial t} + d_2 u_t \wedge dt$$

$$\alpha \wedge d\alpha = \underbrace{\cancel{\beta_t \wedge d_2\beta_t}}_{0 \text{ since 3-form on } \Sigma} + \beta_t \wedge dt \wedge \frac{\partial \beta_t}{\partial t} + \beta_t \wedge d_2 u_t \wedge dt + u_t\, dt \wedge d_2\beta_t$$

$$= \left(\beta_t \wedge d_2 u_t + u_t\, d_2\beta_t - \beta_t \wedge \frac{\partial \beta_t}{\partial t}\right) \wedge dt > 0$$

let x be a singularity of Σ_ξ (i.e. zero of β_0)

then at x $\quad u_0\, d_2\beta_0 \wedge dt > 0$

so $d\beta_0 \neq 0$ near x on Σ

so $d\beta_0$ is an area form near x

so $\exists$ a vector field v s.t. $\iota_v d\beta_0 = \beta_0$

(recall an area form gives an isomorphism $\omega: \mathfrak{X}(\Sigma) \to \Omega^1(\Sigma): v \mapsto \iota_v \omega$)

so $d\iota_v d\beta_0 = d\beta_0$ and thus $\mathrm{div}_{d\beta_0}(v)$ at $x = 1 \underline{\neq 0}$ ✓

($\Leftarrow$)

if $\alpha = \beta_t + u_t\, dt$ is any 1-form on $\Sigma \times [-1,1]$

then it is a contact form

$$\Longleftrightarrow$$

$$u_t\, d\beta_t + \beta_t \wedge \left(du_t - \frac{\partial \beta_t}{\partial t}\right) \neq 0$$

we are given $\mathcal{F}$

let ω be an area form on Σ (given correct or^n)

suppose $\mathcal{F}$ is orientable (so $\mathcal{F}$ = flow of some vector field v)

let $\beta = \iota_v \omega$

let u be the function on Σ s.t. $d\beta = u\omega$

Claim: there is a 1-form γ on Σ s.t. $\gamma \wedge \beta \geq 0$
and $\gamma \wedge \beta > 0$ away from zeros of β

given this set $\beta_t = \beta + t(du + \gamma)$

and $\alpha = \beta_t + u\,dt$

we compute $d\alpha = dt \wedge \frac{\partial \beta_t}{\partial t} + d\beta_t + du \wedge dt$

as before

$$\text{so } \alpha \wedge d\alpha = \left(\beta_t \wedge d\beta_t - \beta_t \wedge \frac{\partial \beta_t}{\partial t} + \beta_t \wedge du \wedge dt + u\, d\beta_t\right) \wedge dt$$

$$= \left[\left(\beta + t(du + \gamma)\right) \wedge \left(-du - \gamma + du\right) + u\left(d\beta + t\,d\gamma\right)\right] \wedge dt$$

$$= \left(\gamma \wedge \beta - t\,du \wedge \gamma + u^2 \omega + tu\,d\gamma\right) \wedge dt$$

$$= \left(\gamma \wedge \beta + u^2 \omega\right) \wedge dt$$

at $t=0$

note: u is the divergence of v

so $u \neq 0$ at singular points of $\mathcal{F}$

$\therefore$ $\alpha \wedge d\alpha > 0$ there

at non singular points $\gamma \wedge \beta > 0$ and $u^2 \geq 0$

so $\alpha \wedge d\alpha > 0$ there too

$\therefore$ α a contact form for t near zero.

now for the proof of the claim

exercise: show there is an almost complex structure J on Σ such that $\omega(w, Jw) > 0$ $\forall w \neq 0$

given this let $\gamma = \iota_{-Jv}\,\omega$

note $\gamma \wedge \beta(v, Jv) = \gamma(v)\beta(Jv) - \gamma(Jv)\beta(v)$

$$= \omega(-Jv, v)\, \omega(v, Jv) - \omega\cancelto{0}{(Jv, Jv)}\, \omega\cancelto{0}{(v, v)}$$

$$= (\omega(v, Jv))^2 \geq 0 \quad \text{and } > 0 \text{ if } v \neq 0$$

so v has desired property

if ξ not orientable, then can work in a cover

exercise: check this case

Prop 2:

let P be a C^∞ generic property of vector fields

let Σ be a surface embedded in a contact manifold (M, ξ)

By a C^∞ small isotopy of Σ we can arrange for Σ_ξ to satisfy P (i.e. Σ_ξ flow of vector field satisfying P)

Proof: let $\alpha = \beta_t + u_t\, dt$ be a contact form on a neighborhood $N = \Sigma \times [-1, 1]$ of Σ in M

let ω be an area form on Σ

since $\mathcal{X}(\Sigma) \to \Omega^1(\Sigma) : v \mapsto \iota_v \omega$ is an isomorphism

$\exists$ vector field v_t such that $\iota_{v_t}\omega = \beta_t$

perturb v_t to v_t' s.t. v_0' satisfies P

let $\beta_t' = \iota_{v_t'}\, \omega$ and

$\alpha' = \beta_t' + u_t\, dt$

if v_t' close to β_t', then α' close to α

and since $\alpha \wedge d\alpha > 0$ is an open condition

α' is a contact form

and so is $\alpha_s = s\alpha' + (1-s)\alpha$

so Gray's Th^m (Th^m II. 6) gives an isotopy

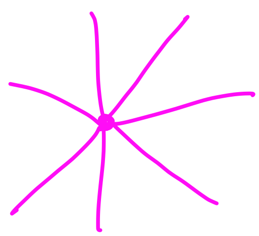
$\phi_s : M \to M$ s.t. $\phi_s^* \alpha_s = \alpha_0$

since $\Sigma_{\xi' = \ker \alpha'}$ is given by β_0' it

satisfies P and $\phi_s^{-1}(\Sigma)$ is an isotopy

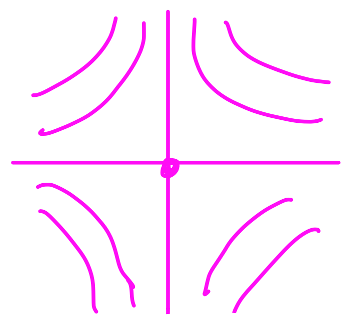
of Σ to $\phi_1^{-1}(\Sigma)$

a generic vector field has isolated zeros and they are either



node (or elliptic)

or



saddle (or hyperbolic)

call a singular point <u>positive</u> if $\operatorname{div}_\omega(v) > 0$

<u>negative</u> if $\operatorname{div}_\omega(v) < 0$

for any area form ω inducing orientation on Σ

and any vector field v directing Σ_ξ

<u>note</u>: this is equivalent to whether or not the orientation

on ξ and on $T\Sigma$ agree at the singularity

indeed if $\alpha = \beta_t + u_t\, dt$, then at a singular point x

$$\alpha \wedge d\alpha = u_t\, d\beta_t \wedge dt > 0$$

(since $\beta_t = 0$ at x)

so $u_0\, d\beta_0$ is a positive volume form on Σ near x

let v be a vector field s.t.

$$\iota_v\, u_0\, d\beta_0 = \beta_0$$

we have $\quad d\iota_v\, u_0\, d\beta_0 = d\beta_0$

$$\text{so } \operatorname{div}_{u_0 d\beta_0} v = \frac{1}{u_0}$$

so if divergence >0 then $u_0 > 0$

and $d\beta_0$ induces correct or^{n} on $T_x\Sigma$

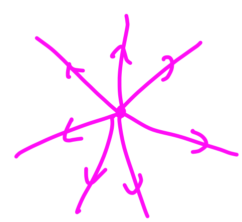
but also induces or^{n} on ξ_x

$\therefore$ div $>0 \Rightarrow +$ singular point

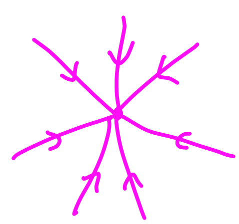
similarly for div <0

exercise: if x is a nodal singular point then

x is positive if



x is negative if



let's think about saddle singular points

consider $\alpha = dz + a\,y\,dx + b\,x\,dy \quad$ for $a, b > 0$

$$\alpha \wedge d\alpha = (dz + a y\, dx + b x\, dy) \wedge (-a\, dx \wedge dy + b\, dx \wedge dy)$$

$$= (b-a)\, dx \wedge dy \wedge dz$$

So α contact $\Leftrightarrow b-a \neq 0$

positive contact $\Leftrightarrow b-a > 0$

if $b-a<0$ then orient $\mathbb{R}^3$ by $dy \wedge dx \wedge dz$

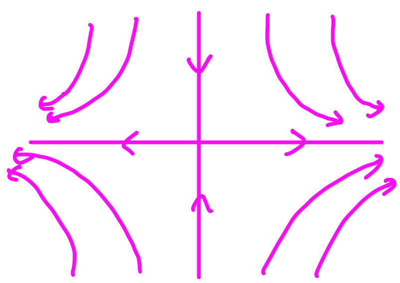
so positive contact str here

xy-plane has a singularity at $(0,0)$

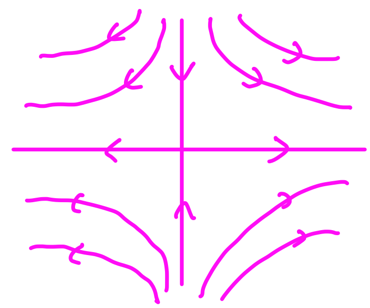
let $\omega = dx \wedge dy$

note $v = \begin{bmatrix} bx \\ -ay \end{bmatrix}$ satisfies $\iota_v \omega = \alpha|_{xy\text{-plane}}$

so characteristic fol^{n} is



pos. singularity



neg. singularity

we end this section with a lemma we need later.

lemma 3:

let L be a Legendrian arc in $\Sigma \subset (M,\xi)$

$x \in L$ an isolated singular point of Σ_ξ

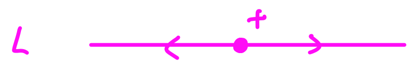
if ξ crosses $T\Sigma$ along L at x in a left-handed way

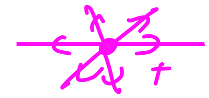
then x is a source (sink) of fol^{n} along L if

the singular point is positive (negative)

if ξ crosses $T\Sigma$ along L at x in a right-handed way
then x is a sink (source) of fol^n along L if
the singular point is positive (negative)

example: left handed twisting we see





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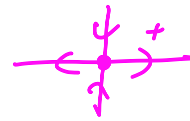
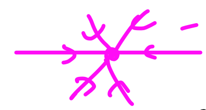


so fol^n could be



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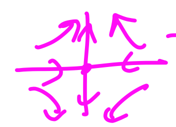
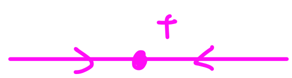
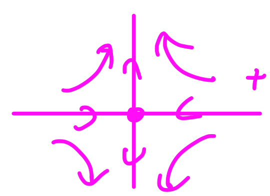
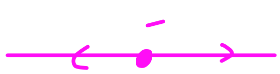


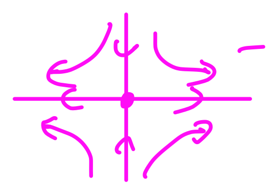
right handed twisting we see





so fol^n <u>must</u> be





Proof: we check left-handed twisting at + singularity

exercise: check other cases

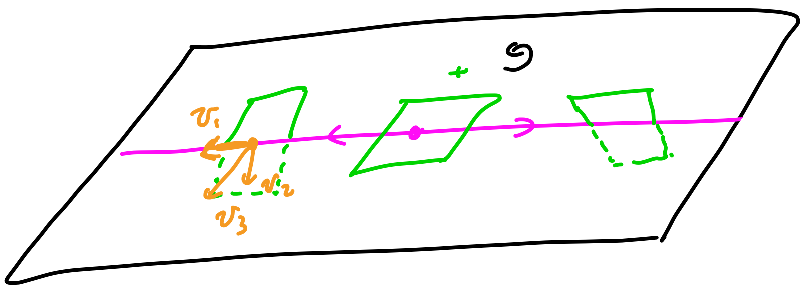
let's recall how to orient $l_x = \xi_x \cap T_x\Sigma$

a vector $v_1 \in l_x$ orients l_x if

there is a vector $v_2 \in \xi_x$, and $v_3 \in T_x\Sigma$

such that v_1, v_2 orients ξ_x, v_1, v_3 orients $T_x\Sigma$

and v_1, v_2, v_3 orients $T_x M$



note v_1, v_2, v_3 orients $T_x M$